Generalized fractional Fourier transforms

1997 J. Phys. A: Math. Gen. 30 973

(http://iopscience.iop.org/0305-4470/30/3/020)

J. Phys. A: Math. Gen. **30** (1997) 973–981. Printed in the UK PII: S0305-4470(97)77994-0

Generalized fractional Fourier transforms

Shutian Liu†‖, Jiuxing Jiang‡, Yan Zhang† and Jingde Zhang§

† Department of Physics, Harbin Institute of Technology, Harbin, 150001, People's Republic of China

‡ Department of Physics, Harbin University of Science and Technology, Harbin, 150080, People's Republic of China

§ Shenzhen Science Hall, Shenzhen, 510831, People's Republic of China

Received 19 September 1996

Abstract. We generalize the definition of the fractional Fourier transform (FRT) by expanding the new definition proposed by Shih to the original definition. The generalized FRT is shown to have k-periodic eigenvalues with respect to the order of Hermite–Gaussian functions and will be reduced to the original FRT and Shih's FRT at the two limits with $k = \infty$ and $k = 4$, respectively. The results of computer simulations and symbolic representations of the transform are given. Properties of the generalized FRT have been discussed.

1. Introduction

Fractional Fourier transforms (FRTs) have recently been demonstrated to be useful in quantum mechanics [1, 2] as well as in optical information processing [3–6]. They expand the conventional Fourier transform to a more general time–frequency (or space–frequency) joint representation [7] by allowing the concept of the fractional integral to be applied in the Fourier integrals. Along with their challenging applications, their mathematical properties still attract more attention. Recently, Shih [8] re-invented a technique to fractionalize the Fourier transform with only the weighted composition of four basic functions, i.e. the original function, its first, second and third Fourier transforms, only according to the three postulates that the FRTs should obey. In this paper, we first discuss the relationship of Shih's FRT with the original definition. We find that Shih's FRT has the same eigenfunction with the conventional Fourier transform, the Hermite–Gaussian polynomials, but with different eigenvalues which have a periodicity of 4 with respect to the order of Hermite polynomials. We then expand Shih's fractionalization procedure allowing the periodicity to be $k = 4l$ $(l = 1, 2, 3, \ldots)$ and demonstrate that Shih's FRT and the original FRT are two extreme cases which correspond to $l = 1$ and $l = \infty$, respectively. Computer simulations have demonstrated that the Hermite–Gaussian decomposition algorithms which were used to calculate the original fractional Fourier transform coefficients are still valid for the generalized definition of the FRT only by substituting the eigenvalues with the new ones.

This paper is arranged as follows. In section 2, we briefly introduce Shih's definition of the FRT for the reason that we will reference it frequently in the following context. In section 3, we represent Shih's FRT using the base of Hermite–Gaussian polynomials, and

‖ E-mail address: stliu@fox.hit.edu.cn

then introduce our generalized definition of the FRT by expanding the format of eigenvalues corresponding to the order of Hermite polynomials. Computer simulations are given to compare these two definitions with the original FRT. Some properties of the generalized FRT are briefly discussed in section 4. Finally, we present our conclusions of this paper.

2. Shih's definition of fractional Fourier transform

Unlike the original FRT [1, 2], Shih's definition of the FRT [8] takes an alternative continuous way to fractionalize the Fourier transform. He assumed that any transient state of the Fourier transform could be expressed by its first four integer-order state which is an analogy to quantum states, because the integer-order Fourier transform is a 4-periodic operation. Let the jth-order Fourier transforms $\mathcal{F}^j$, ($j = 0, 1, 2$, and 3) of the original function be $f_0(x)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$, respectively, such that

$$\begin{aligned}
&\mathcal{F}^0\{f_0(x)\} = f_0(x)\\
&\mathcal{F}^1\{f_0(x)\} = f_1(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_0(x)\exp(-\mathrm{i}xx')\,\mathrm{d}x'\\
&\mathcal{F}^2\{f_0(x)\} = \mathcal{F}^1\{f_1(x)\}\\
&\mathcal{F}^3\{f_0(x)\} = \mathcal{F}^1\{f_2(x)\}.
\end{aligned} \tag{1}$$

One can construct the α-order fractional Fourier transform as a weighted composition of these four functions as follows:

$$\mathcal{F}_\mathrm{s}^\alpha\{f(x)\} = A_0^\alpha f_0(x) + A_1^\alpha f_1(x) + A_2^\alpha f_2(x) + A_3^\alpha f_3(x) \tag{2}$$

where the weight coefficients, A_j^α, $j = 0, 1, 2$, and 3, are continuous functions of the fractional order only. Here we denote Shih's FRT operator by $\mathcal{F}_\mathrm{s}^\alpha$ in order to distinguish it from the original FRT $\mathcal{F}^\alpha$. Such defined FRTs obey the following postulates:

(1) $\mathcal{F}_\mathrm{s}^\alpha$ should be continuous for all real values α;

(2) $\mathcal{F}_\mathrm{s}^\alpha$ should reduce to an ordinary Fourier transform when α is an integer, i.e. $\mathcal{F}_\mathrm{s}^1 = f_1(x)$;

(3) $\mathcal{F}_\mathrm{s}^\alpha$ should have the additive property:

$$\mathcal{F}_\mathrm{s}^{\alpha+\beta}\{f(x)\} = \mathcal{F}_\mathrm{s}^\alpha\{\mathcal{F}_\mathrm{s}^\beta\{f(x)\}\} = \mathcal{F}_\mathrm{s}^\beta\{\mathcal{F}_\mathrm{s}^\alpha\{f(x)\}\}. \tag{3}$$

Note that the above three postulates should be satisfied by all definitions of the FRT including the original one. Using these properties, one can obtain a set of coupled equations for the coefficients as follows:

$$\begin{aligned}
A_0^{\alpha+\beta} &= A_0^\alpha A_0^\beta + A_1^\alpha A_3^\beta + A_2^\alpha A_2^\beta + A_3^\alpha A_1^\beta\\
A_1^{\alpha+\beta} &= A_0^\alpha A_1^\beta + A_1^\alpha A_0^\beta + A_2^\alpha A_3^\beta + A_3^\alpha A_2^\beta\\
A_2^{\alpha+\beta} &= A_0^\alpha A_2^\beta + A_1^\alpha A_1^\beta + A_2^\alpha A_0^\beta + A_3^\alpha A_3^\beta\\
A_3^{\alpha+\beta} &= A_0^\alpha A_3^\beta + A_1^\alpha A_2^\beta + A_2^\alpha A_1^\beta + A_3^\alpha A_0^\beta.
\end{aligned} \tag{4}$$

The coefficients A_j^α are solved to be

$$A_j^\alpha = \exp\left(\mathrm{i}3\pi\frac{\alpha-j}{4}\right)\cos\left(\pi\frac{\alpha-j}{2}\right)\cos\left(\pi\frac{\alpha-j}{4}\right). \tag{5}$$

Symbolically, Shih's FRT can be more compactly written as

$$\mathcal{F}_\mathrm{s}^\alpha\{f(x)\} = \sum_{j=0}^{3}\exp\left(\mathrm{i}3\pi\frac{\alpha-j}{4}\right)\cos\left(\pi\frac{\alpha-j}{2}\right)\cos\left(\pi\frac{\alpha-j}{4}\right)f_j(x). \tag{6}$$

3. Generalized definition of fractional Fourier transform

The first question we may ask is: What kind of relationship exists between Shih's definition of the FRT and the original one? Does Shih's definition represent a general description of all transient states between the original image and its Fourier transform? Obviously Shih's FRT is different from the original FRT which can be represented as an integral [2]

$$\mathcal{F}^{\alpha}\{f(x')\} = \sqrt{\frac{1-\mathrm{i}\cot(\phi_\alpha)}{2\pi}}\exp[\mathrm{i}x^2\cot(\phi_\alpha)] \times \int_{-\infty}^{\infty} f(x')\exp\left[\mathrm{i}\left(x'^2\cot(\phi_\alpha) - 2xx'\frac{1}{\sin(\phi_\alpha)}\right)\right]\mathrm{d}x' \tag{7}$$

because Shih's FRT cannot be written in such an integration format. Here $\phi_\alpha = \pi\alpha/2$ denotes an angle corresponding to the fractional order α. In our logical reasoning, Shih's FRT is only a special case in the processes of the fractionalization of the Fourier transform. There is no unique approach for the evolution from an image to its Fourier spectrum. Furthermore, the original FRT represents transient states between the integer-order Fourier transforms which in optics is related frequently with Fresnel diffraction; however, the original FRT cannot be represented as a weighted combination of the first four integer-order Fourier transforms.

In the following context we will discuss in what way Shih's FRT is related to the original FRT. Because Shih's FRT is combined by the first four integer-order Fourier transforms of the original function, it is reasonable to believe that the FRT has the same eigenfunction as the conventional Fourier transform, i.e. the Hermite–Gaussian function

$$\Psi_m(x) = \exp\left(-\frac{x^2}{2}\right)H_m(x) \tag{8}$$

where $H_m(x)$ is the mth-order Hermite polynomial having the recursive relation

$$H_{m+1}(x) = 2xH_m(x) - 2nH_{m-1}(x). \tag{9}$$

It is well known that the jth order Fourier transform operator has an eigenvalue $\exp(\mathrm{i}\pi mj/2)$. Therefore, one can calculate the eigenvalues for Shih's FRT from equation (2). Assuming $m = 4n + k$, where n is any integer and $k = 0, 1, 2$, and 3, substituting the function $f_0(x)$ with $\Psi_m(x)$, one can obtain

$$\mathcal{F}_{\mathrm{s}}^{\alpha}[\Psi_m(x)] = \exp\left(\frac{\mathrm{i}\pi k\alpha}{2}\right)\Psi_m(x) = \lambda_m\Psi_m(x). \tag{10}$$

The eigenvalues λ_m for fractional Fourier operator $\mathcal{F}_{\mathrm{s}}^{\alpha}$ can be found as

$$\lambda_m = \exp\left(\frac{\mathrm{i}\pi k\alpha}{2}\right) \qquad k = \mathrm{mod}(m, 4) \tag{11}$$

where k is the remainder of m divided by 4. Therefore, Shih's assumption mandated another periodicity of 4. As one can see, the eigenvalues λ_m are 4-periodic not only with respect to the order of the transform α but also to the order of the Hermite functions m. One should note that the eigenvalues for the original FRT are $\lambda_m = \exp(\mathrm{i}\pi\alpha m/2)$, which have no periodicity with respect to the order of Hermite polynomials. This fact may be regarded as the distinction between these two definitions.

From the derived eigenvalues, $\mathcal{F}_{\mathrm{s}}^{\alpha}[f(x)]$ can then be expressed by the superposition of Hermite–Gaussian functions,

$$\mathcal{F}_{\mathrm{s}}^{\alpha}[f(x)] = \sum_m A_m\lambda_m\psi_m(x) = \sum_m A_m\psi_m(x)\exp\left(\frac{\mathrm{i}\pi\,\mathrm{mod}(m,4)\alpha}{2}\right) \tag{12}$$

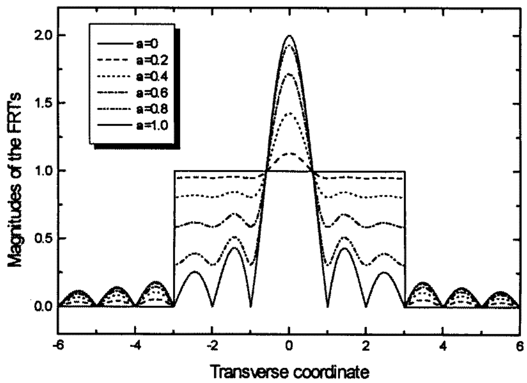

Figure 1. Shih's fractional Fourier transforms of a rectangular function calculated according to equation (2).

where $\psi_m(x) = (1/\sqrt{2^m\sqrt{\pi}m!})H_m(x)\exp(-x^2/2)$ are the normalized Hermite–Gaussian functions and $A_m = \int_{-\infty}^{\infty} f(x)\psi_m(x)\,\mathrm{d}x$ are the superposition coefficients of the function $f(x)$ onto the base of $\psi_m(x)$.

Shih's definition of FRT leads to a fundamental issue that there may exist infinite approaches to the fractionalization of the Fourier transform. Following the three postulates mentioned above, one can fractionalize the Fourier transform by expanding the periodicity, with respect to the orders of the Hermite–Gaussian functions, from 4 to the more general case k, with $k = 4l$ and $l = 1, 2, 3, \ldots$ positive integers. The expanded definition of the FRT can thus be written as

$$\mathcal{F}_k^{\alpha}[f(x)] = \sum_{m=0}^{\infty} \exp\left[\mathrm{i}\frac{\pi}{2}\alpha \mathrm{mod}(m, k)\right] A_m\psi_m(x) \tag{13}$$

where we denote the k-periodic expanded FRT as $\mathcal{F}_k^{\alpha}$. The reason that k must be a multiple of 4 is due to the confinement of the three postulates. It is not difficult to prove that equation (13) satisfies these conditions.

The generalized FRT serves as a bridge to connect Shih's FRT and the original definition. From equation (13) one can easily recognize that these two definitions are just two extreme cases of the expanded definition with $k = 4$ and $k = \infty$, respectively.

Shih's definition of the FRT consists of the weighted sum of the first four Fourier transforms of a function. However, for the k-periodic presentation, the four integer-order Fourier transforms are not enough to construct the fractional Fourier transform. Besides the integer-order Fourier transforms, we must insert more intermediate states of Fourier transforms $\mathcal{F}^{\beta}[f(x)]$, $\beta = 4n/k$, $n = 0, 1, \ldots, k-1$, between the zeroth- and fourth-order Fourier transforms. We must realize that they are the FRTs under the original definition [2]. The superposition can be expressed as

$$\mathcal{F}^{\alpha}[f(x)] = \sum_{n=0}^{k-1} B_n\mathcal{F}^{\beta}[f(x)] \tag{14}$$

where B_n is the decomposition coefficients which will be given below. Using the superposition of Hermite–Gaussian polynomials, equation (17) becomes

$$\sum_{n=0}^{k-1} B_n \sum_{m=0}^{\infty} \exp\left(\mathrm{i}\frac{\pi}{2}\frac{4mn}{k}\right) A_m\psi_m(x) = \sum_{m=0}^{\infty} \exp\left[\mathrm{i}\frac{\pi}{2}\alpha \mathrm{mod}(m, k)\right] A_m\psi_m(x). \tag{15}$$

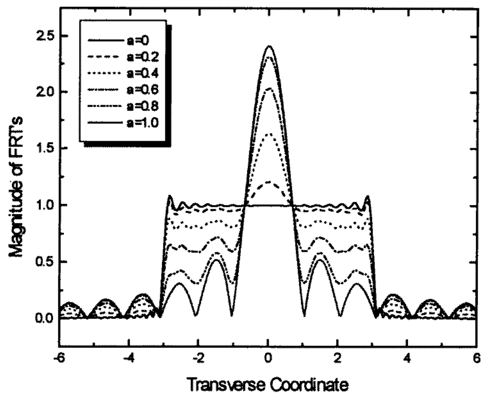

Figure 2. Shih's fractional Fourier transforms of a rectangular function calculated from the Hermite–Gaussian superposition algorithm. In our calculation, the first 200 Hermite polynomials were employed.

This means that

$$\sum_{n=0}^{k-1} B_n \exp\left(\mathrm{i}\frac{\pi}{2}\frac{4mn}{k}\right) = \exp\left[\mathrm{i}\frac{\pi}{2}\alpha \mathrm{mod}(m,k)\right] \tag{16}$$

which is equivalent to

$$B_n = \sum_{m=0}^{k-1} \frac{1}{k} \exp\left(-\mathrm{i}2\pi\frac{mn}{k}\right)\exp\left(\mathrm{i}\frac{\pi m\alpha}{2}\right). \tag{17}$$

Thus we have

$$B_n = \frac{1}{k}\frac{1-\exp[\mathrm{i}2\pi(\alpha l - n)]}{1-\exp[\mathrm{i}2\pi(\alpha l - n)/k]} = \frac{1}{k}\exp\left[\mathrm{i}\pi(\alpha l - n)\frac{k-1}{k}\right]\frac{\sin[\pi(\alpha l - n)]}{\sin[\pi(\alpha l - n)/k]} \tag{18}$$

where one should remember that $k = 4l$ and $\beta = 4n/k = n/l$. The final result will be

$$\mathcal{F}_k^{\alpha}[f(x)] = \sum_{n=0}^{k-1}\frac{1}{k}\exp\left[\mathrm{i}\pi(\alpha l - n)\frac{k-1}{k}\right]\frac{\sin[\pi(\alpha l - n)]}{\sin[\pi(\alpha l - n)/k]}\mathcal{F}^{\beta}[f(x)]. \tag{19}$$

This equation will be naturally reduced to Shih's FRT with $k = 4$, i.e. $l = 1$. When $k \to \infty$, the coefficient

$$\frac{1}{k}\exp\left[\mathrm{i}\pi(\alpha l - n)\frac{k-1}{k}\right]\frac{\sin[\pi(\alpha l - n)]}{\sin[\pi(\alpha l - n)/k]} \tag{20}$$

turns out to be a Kronecker delta function $\delta(\alpha l, n)$. One can, therefore, write the case of $k \to \infty$ as

$$\mathcal{F}_k^{\alpha}[f(x)] = \mathcal{F}^{\alpha}[f(x)]. \tag{21}$$

4. Properties of the generalized fractional Fourier transform

We have computer-simulated Shih's FRT and the generalized FRT on a rectangular function and compared the results with the plots generated by the original FRT. Figures 1 and 2 present Shih's FRT by two different approaches from equation (2) and equation (12),

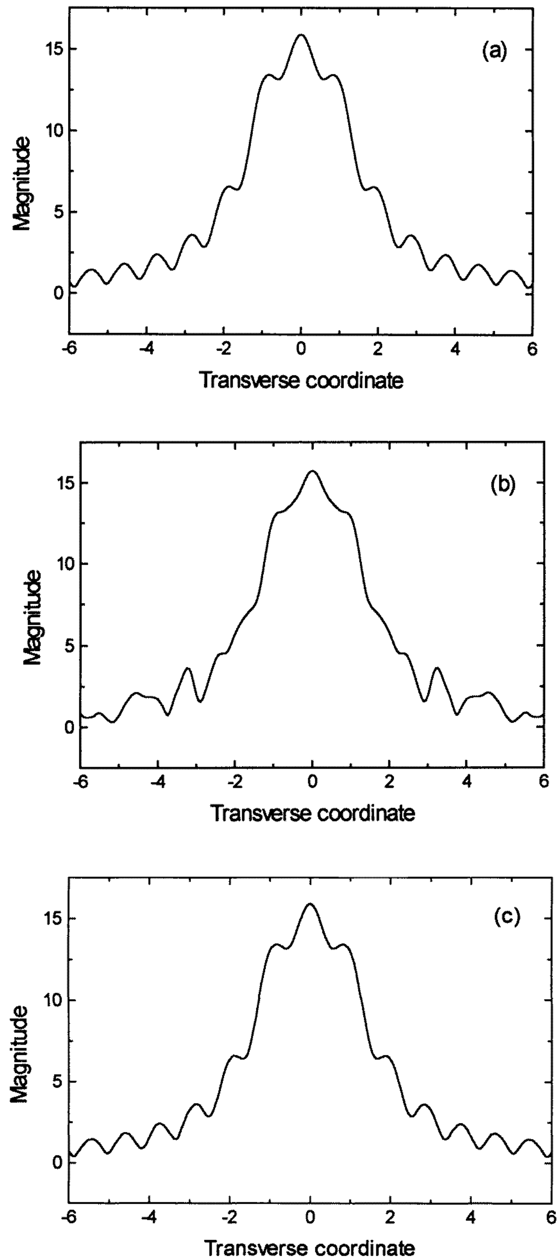

Figure 3. The fractional Fourier transform of a rectangular function, $\alpha = 0.6$, calculated according to (a) the original definition, (b) and (c) the generalized definition of the FRT with $k = 16$ and $k = 20$, respectively.

respectively. The results show that these two equations are identical unless in figure 2 we only use the first $N = 200$ Hermite–Gaussian functions and thus the original function cannot be recovered completely. Figure 3 gives the simulation results of the generalized FRT and the comparison with the original FRT. Figure 3(a) shows the FRT of order $\alpha = 0.6$

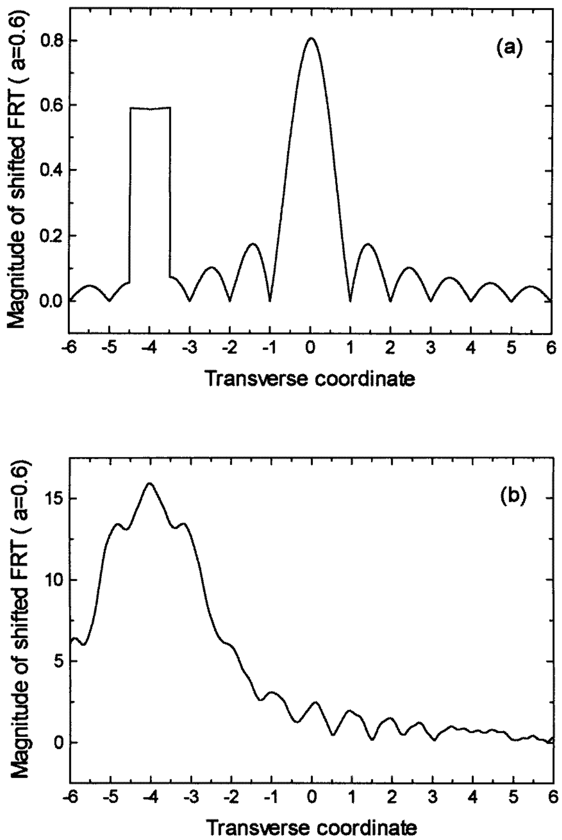

Figure 4. The shift property of the generalized FRT for (a) $k = 4$ and (b) $k = 64$. The rectangular function $f(x)$ is shifted to $f(x+4)$.

while figure 3(b) indicates the result of the generalized FRT with the same order but $k = 16$. The plot in figure 3(b) is similar to that of figure 3(a). When the periodicity k becomes larger, these two plots are identical.

Furthermore, for any given positive integer l, there are $l - 1$ generalized FRTs with fractional orders $\alpha = 1/l, 2/l, \ldots, l - 1/l$, $\alpha \in [0, 1]$, which are identical to the original FRTs with the same orders. For example, in the case of $l = 2$, $\mathcal{F}_k^{0.5}[f(x)] \equiv \mathcal{F}^{0.5}[f(x)]$. If $l \to \infty$, the relation $\mathcal{F}_k^{\alpha}[f(x)] \equiv \mathcal{F}^{\alpha}[f(x)]$ always holds for any orders. The special case is that of Shih's definition which is identical to the original FRT only at integer orders. Figure 3(c) shows the result of the generalized FRT with $k = 20$ and $\alpha = 0.6$, which is identical to the curve in figure 3(a).

All the three kinds of FRT are scale- and shift-variant transforms. This property can be easily seen from the computer simulations. During the processes of the fractionalization from Shih's FRT to the original FRT by quadrupling the periodicity k, the spatial (or temporal) and frequency information of the input image becomes more and more mixed, which means that at lower periodicity k the FRT has a looser space–frequency representation. Figure 4(a) shows the simulation of a shifted Shih's FRT of a rectangular function with $\alpha = 0.6$. The shifting parameter is $b = 4$. When the shift b is sufficiently large, the FRT spectrum consists both of the spatial and frequency information of the original input in which the frequency part can give all the information related to the input image and the spatial part indicates the location of the input. For some space-variant applications this may be a nice property. However, when k grows, these two pieces of information mix together. Figure 4(b) represents the shifted generalized FRT with $k = 64$ with the same condition as for figure 4(a).

An interesting property associated with the generalized FRT concerns the concept of self-fractional-Fourier functions which originate from Caola's self-Fourier functions [9]. Similar to the meaning of self-Fourier function, a function is called a self-fractional-Fourier function if it transforms to itself after undergoing a fractional Fourier transform. Mendlovic *et al* [10] have given a construction of a self-fractional-Fourier function $F^{\alpha}(x)$ via any transformable function $f(x)$

$$F^{\alpha}(x) = \tilde{f}^{0}(x) + \tilde{f}^{\alpha}(x) + \tilde{f}^{2\alpha}(x) + \cdots + \tilde{f}^{(k'-1)\alpha}(x) \tag{22}$$

where $\tilde{f}^{\alpha}(x)$ is the α-order fractional Fourier transform of the function $f(x)$, while k' and l' are the minimum integers which satisfy the relation $k' = 4l'/\alpha$. When $\alpha = 1$, then $k' = 4$, so that the function $F^{\alpha}(x)$ turns to the self-Fourier function

$$F(x) = f_0(x) + f_1(x) + f_2(x) + f_3(x) \tag{23}$$

with the same meanings here; $f_0(x)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$ are the first four integer-order Fourier transforms of function $f(x)$.

Let $k = k' = 4l$, and then we have $l = l'/\alpha$. Using this relation, one can analyse the characteristics of self-fractional-Fourier functions under the three definitions of FRT. For Shih's case, $l = 1$, which forces the order $\alpha = 1$ in order to have a minimum integer $l' = 1$. This means that the self-Fourier function is unchanged after experiencing a Shih's FRT. It can be understood that a self-Fourier function $F(x)$ has a set of 4-periodic eigenvalues respective to the Hermite–Gaussian functions $\{4, 0, 0, 0\}$. Similarly, functions $\mathrm{i}F(x)$, $-F(x)$, and $-\mathrm{i}F(x)$ which correspond to the eigenvalues of $\{0, 4, 0, 0\}$, $\{0, 0, 4, 0\}$, and $\{0, 0, 0, 4\}$, respectively [11], are also unchanged after Shih's FRT.

For the generalized FRT and $k > 4$, the self-Fourier function cannot be unchanged; however, we can obtain the self fractional Fourier functions of orders $\alpha = (l - l')/l, (l - l' - 1)/l, \ldots, (l - 1)/l$, respectively. For example, if $k = 16$, one can construct three self-fractional-Fourier functions: $F^{\frac{1}{4}}(x)$, $F^{\frac{1}{2}}(x)$, and $F^{\frac{3}{4}}(x)$. For a conventional FRT, there exists a self-fractional-Fourier function for any given fractional order $\alpha \neq 1$.

5. Conclusions

We have generalized Shih's definition of the fractional Fourier transform by taking a k-periodic eigenvalue with respect to the orders of the Hermite–Gaussian base. The generalized FRT naturally links the Shih's definition of the FRT and the original FRT in the limitations of $k = 4$ and $k = \infty$, respectively. The expressions of superposition on Hermite–Gaussian functions and on the original fractional Fourier transforms for the generalized FRT are given. Its property concerned with the self-fractional-Fourier functions are discussed.

Acknowledgments

This research is supported by the Natural Science Foundation of China under contract No 69577006, the Natural Science Foundation of Heilongjiang Province and the Research Foundation of the Education Committee of China.

References

[1] Namias V 1980 The fractional order Fourier transform and its application to quantum mechanics *J. Inst. Math. Appl.* **25** 241–65
[2] McBride A C and Kerr F H 1987 On Namias' fractional Fourier transforms *IMA J. Appl. Math.* **39** 159–75
[3] Mendlovic D and Ozaktas H M 1993 Fractional Fourier transformations and their optical implementation: part II *J. Opt. Soc. Am.* A **10** 2522–31
[4] Lohmann A W 1993 Image rotation, Wigner rotation, and the fractional Fourier transform *J. Opt. Soc. Am.* A **10** 2181–6
[5] Ozaktas H M, Barshan B, Mendlovic D and Onural L 1994 Convolution, filtering, and multiplexing in fractional Fourier domain and their relation to chirp and wavelet transform *J. Opt. Soc. Am.* A **11** 547–59
[6] Mendlovic D, Ozaktas H M and Lohmann A W 1995 Fractional correlation *Appl. Opt.* **34** 303–9
[7] Almeida L B 1994 The fractional Fourier transform and time-frequency representations *IEEE Trans. Sign. Proc.* **42** 3084–91
[8] Shih C C 1995 Fractionalization of Fourier transform *Opt. Commun.* **118** 495–8
[9] Caola M J 1991 Self-Fourier functions *J. Phys. A: Math. Gen.* **24** L1143–4
[10] Mendlovic D, Ozaktas H M and Lohmann A W 1994 Self-Fourier function and fractional Fourier transform *Opt. Commun.* **151** 36–8
[11] Cincotti G, Gori F and Santarsiero M 1992 Generalized self-Fourier functions *J. Phys. A: Math. Gen.* **25** L1191–4